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Mathematical modelling of mountain height distribution on the Earth's surface

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Analysis of the distribution of the Earth's highest mountains shows that the Earth's surface can be modelled by a mathematical surface which is more complicated than a usual fractal and the dimension of which is not a constant value. The deviations of the obtained approximating curve of the mountains' height from the actual height are shown to represent a statistical noise close to $1 / f^2$. The total number of mountains and the maximum possible height of a mountain on the Earth are assessed. The concept of the distribution density of mountains (orosity) is introduced, which may be useful in economic assessments.

Key words: mountains, fractals, noise $1 / f^2$, mathematical modelling

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INTRODUCTION

It has been known long since that a small piece of rock by its form is very similar to the mountain from which it originates. For instance, Edward Whymper in his "Scrambles amongst the Alps" writes: "It is worthy of remark that … fragments of rock … often present the characteristic forms of the cliffs from which they have been broken".

However, to what extent this similarity can be characterized quantitatively? This can be studied by using the mathematical fractal introduced by Benoit Mandelbrot and characterized by two basic properties – self-similarity and fractal dimension (Mandelbrot, 1983). Summarizing L. F. Richardson's work (Ashford, 1993), B. Mandelbrot showed that the coastal line of Great Britain and some other seashore lines may be modelled by fractal curves. B. Mandelbrot owns the idea that the whole Earth's surface may be modelled by fractal multitudes. Indeed, fractal surfaces drawn by computer (Mandelbrot, 1983) bear a strong resemblance to mountain-masses.

The further applications of fractals in this area are related to fractal investigations of geological media (Ivaniuk, 1997), auto-modelling of geodynamic processes (Sadovskij, 1986), application of fractal structures in seismic studies (Barriere, Turcotte, 1997).

The paper presents a natural approach instead of mathematical one. Let us analyze the list of the Earth-highest mountains and determine how much it may be related to fractal geometry.

The scheme of the paper is as follows. First of all, we compile and verify a list of the 548 highest mountains of the continents, i. e. those higher than 3 500 m. The next two parts deal with (1) the mathematical function that approximates the heights of mountains and (2) analyses the deviation of the observed heights from the theoretical heights. Then we select the number of mountains with the height no less than *h*. In the fifth part, the term of the distribution density of mountains (orosity) (from the Greek *oros* – mountain) is introduced, and in the sixth the possible maximum height of a mountain is assessed. The seventh part presents a comparison of exponential and power approximations.

MATERIALS AND METHODS

Approximation

As a source of information, a list compiled by P. Scaruffi (Scaruffi, 2008) was used. It comprises 548 mountains higher than 3 500 m.

In the list, the distribution of the mountains (Fig. 1a) is not very regular. Its only regular feature is the decrease of the height h_n in the sequence of mountains.

Let us take, instead of sequence h_n, the sequence $ln\ (h_1 / h_n)$, (h_1 = 8 848) and represent it in a double logarithmic scale (Fig. 1b).

This dependence looks more regular. Mountain height logarithms are well approximated by the linear function $\alpha x + \beta$ in which the dimensionless coefficients are

$$\alpha = 0.54044; \quad \beta = 3.1170 \cdot 10^{-2}. \tag{1}$$

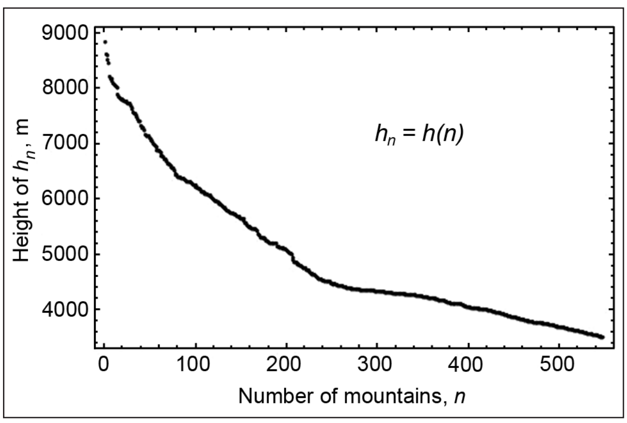

Fig. 1a. Distribution of highest mountains worldwide ($h_n \geq$ 3 500 m)

1 pav. Aukščiausių kalnų pasiskirstymas Žemėje

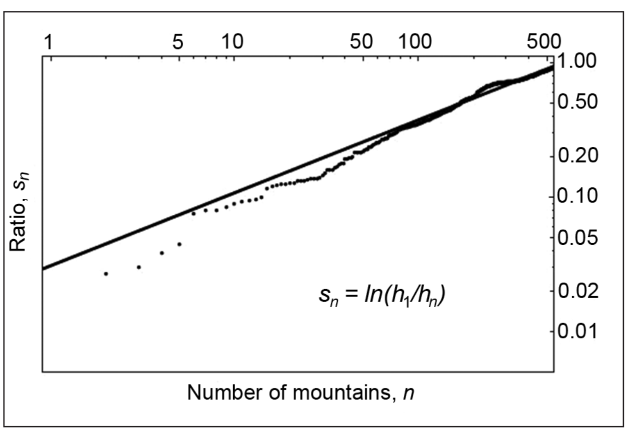

Fig. 1b. Logarithm distribution of heights of highest mountains in a double logarithmic scale

1b pav. Aukščiausių kalnų aukščio pasiskirstymas dviguboje logaritminėje skalėje

The initial dependence of mountain height distribution can be approximated by the function

$$h(x) = h_1 e^{-\beta x^{\alpha}}, \quad h_1 = 8\,848\ m. \tag{2}$$

Indeed, summation of dependence h_n and the approximating function $h\ (x)$ gives a rather good correspondence (Fig. 2a).

The obtained approximating function (2) allows determining the height of the nth mountain. Extrapolation of this formula for an unknown region allows determining the height of, e. g., the 600th or 700th mountain as h (600) = 3 291 m and h (700) = 3 020 m, respectively.

Deviations

For a quantitative assessment of the obtained approximation and of errors while extrapolating formula (2) into an unknown region, we shall find the deviations of the obtained function from the real height of the mountains. We shall see the deviation δh_n as the difference

$$\delta h_n = h_n - h(n), \quad n = 1, 2, \ldots \tag{3}$$

The dependence δh_n of real deviations is shown in Fig. 2b. The same figure shows also the deviation interval: the mean square deviation (standard deviation) δh = 155.5 ≈ 156 m from the mean value $<\delta h>$ = –19.8 m. We see that most of the deviation values fit within the interval of two mean square deviations.

Like the initial mountain height distribution h_n, the deviation dependence δh_n does not look very regular. The dependence is neither monotonous nor periodic. In case some periodicity is present in this dependence, it should be manifested in its Fourier spectrum. Figure 3 shows the $|a_m|$ spectrum of dependence δh_n:

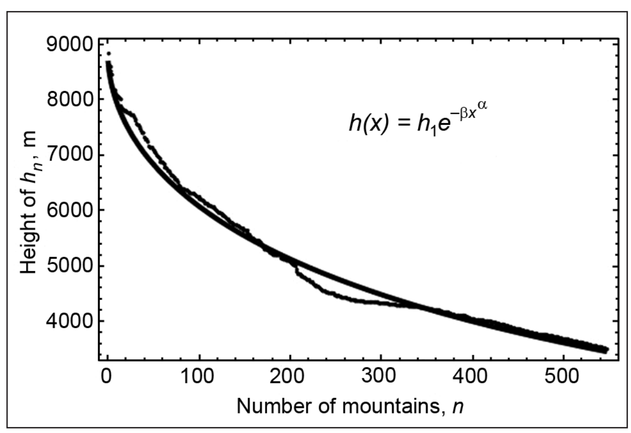

Fig. 2a. Distribution of world's highest mountains ($h \geq$ 3 500 m) and the approximating theoretical function

2a pav. Aukščiausių žemės kalnų ($h \geq$ 3 500 m) pasiskirstymas ir aproksimuojanti teorinė funkcija

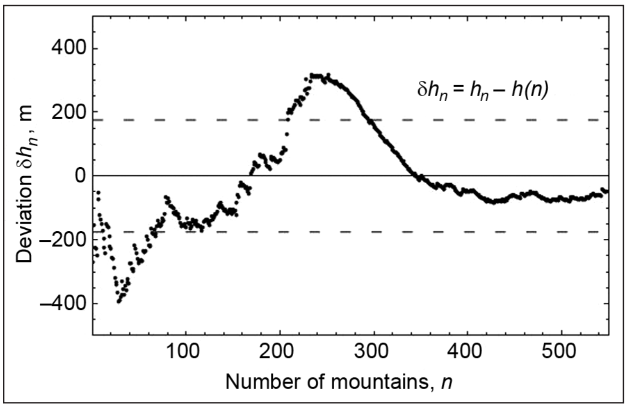

Fig. 2b. Deviations of approximated mountain heights from observed heights and the mean square deviation $\delta h_n = 156$ m

2b pav. Kalnų aproksimuotų aukščių nukrypimai nuo esamų aukščių ir vidurkinis kvadratinis nuokrypis $\delta h_n = 156$ m

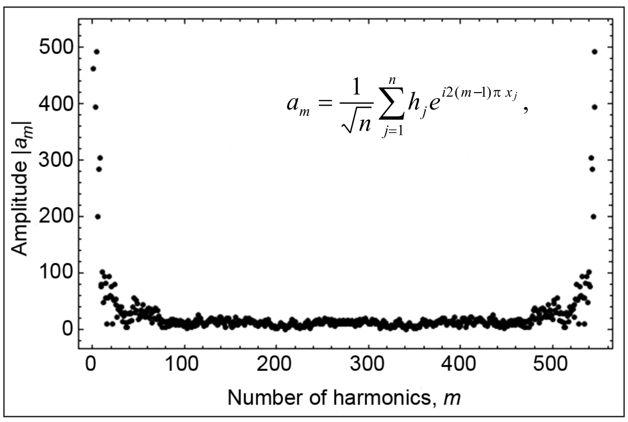

Fig. 3. The Fourier spectrum of the deviations

3 pav. Nuokrypių Furje spektras

$$a_m = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} h_j e^{i2(m-1)\pi x_j},$$
$$\text{here } x_j = \frac{j-1}{n}. \tag{4}$$

We see that the deviation dependence δh_n is not periodic but is similar to a noise described by the spectral density function $S(f) \infty 1 / f^{\alpha}$.

To verify this hypothesis, we shall decrease the number of harmonics from 548 to 500 and perform 10 casual Fourier transformations (this may be done in 48 independent ways). Let us average over these 10 transformations and limit ourselves to the first 50 members. As follows from this averaging procedure, the dependence $|a_m|$ may be approximated by the function

$$|a_m| \approx \frac{1.613 \cdot 10^3}{f^{0.9837}}. \tag{5}$$

The spectral density function $S(f) \infty |a_m|^2$, so we see that our hypothesis has come true: the mountain height dependence δh_n may be considered as noise $S(f) \infty 1 / f^{1.967}$.

This type of deviations, in author's opinion, is not incidental. The noise $1 / f$ for $\alpha = 1$ is known to manifest itself in various branches of science such as physics, biology, finances. Although at present the nature of this noise is not clear, the universal character of the phenomenon is related to the properties of fractal multitudes (Mandelbrot, 1999). However, in our case, noise $S(f) \infty 1 / f^{1.967}$ is close to $1 / f^2$, indicating that neither the deviations nor the initial distribution of heights are a fractal multitude.

RESULTS AND DISCUSSION

The number of mountains of height no less than *h*

From the same theoretical formula (2) it is possible to determine the number of mountains the height of which is no less than h:

$$N = \left\{ \frac{1}{\beta} \ln \frac{h_1}{h} \right\}^{\frac{1}{\alpha}} + 1, \tag{6}$$

here the values of the coefficients α and β are the same as in formula (1), $h_1 = 8\,848$ m. The unity term of formula (6) serves the purpose of numeration: here, the Everest is the mountain number 1 (not number 0). The number N of mountains with the height no less than h is determined within $\delta N_{th} = |N_k|$, i. e.

$$\delta N_{th} = 1.6388 \cdot 10^6 \left\{ \ln \frac{8848}{h} \right\}^{0.85034}. \tag{7}$$

Note here that the error δN strongly depends on h. For instance, the number of mountains higher than 3 500 m is 534 ± 47, i. e. the relative theoretical $\varepsilon_{th} = \delta N_{th} / N_h \cdot 100\%$ is 8.8%, whereas the actual error δN = 548–53 = 4, i. e. even less: ε = 14 / 548 · 100% = 2.6%.

Table 1 presents a list of relative error values also for other heights and, *inter alia*, shows the reliability of the obtained

Table 1. **Theoretical and real numbers of mountains no less than *h* high and their relative errors**

1 lentelė. **Teorinis ir tikrasis skaičius kalnų, ne žemesnių nei *h*, ir jų reliatyvioji paklaida**

***h*, km**	**8.0**	**7.5**	**7.0**	**6.5**	**6.0**	**5.5**	**5.0**	**4.5**	**4.0**	**3.5**
N_{TH}	10	23	43	70	107	156	218	298	401	534
N	14	34	53	77	119	157	206	248	416	548
δN_{TH}	3.1	5.1	7.3	10	13	17	22	28	36	47
δN	4	11	10	7	12	1	12	50	15	14
ε_{TH}	32	22	17	14	12	11	10	9.4	9.0	8.8
ε	30	33	19	8.5	10	0.8	5.8	20	3.7	2.6

theoretical formulae: the mean theoretical relative error $<\varepsilon_{th}>$ = 14.5% exceeds the mean real error $<\varepsilon>$ = 13.3%.

How many mountains are there on the continents?

To answer this question, we must strictly know what we may consider as a mountain. Some authors define a mountain as a peak with a topographic prominence over a definite value. For example, according to the Britannica Student Encyclopedia, the term *mountain* "generally refers to rises over 2.000 feet (610 m)". The Encyclopedia Britannica, on the other hand, does not limit a mountain to any height, merely stating that "the term has no standardized geological meaning".

There is no universally accepted definition of a mountain. Elevation, volume, relief, steepness, spacing and continuity have been used as criteria for defining a mountain (Gerrard, 1990). In the Oxford English Dictionary, a mountain is defined as "a natural elevation of the earth surface rising more or less abruptly from the surrounding level and attaining an altitude which, relatively to the adjacent elevation, is impressive or notable".

In England and Wales, the Department for Environment, Food and Rural Affairs for the purpose of right to roam legislation has defined "a mountain" as all land over 600 meters. The Land and Reform Act 2003 (Scotland) does not appear to draw this distinction, and in Scotland the term "mountain" is more subjective, often being used for hills exceeding 3.000 feet (914.4 m).

We take 600 m, 610 m and 914.4 m as the basis of these three numerical definitions of a mountain. Formulae (6) and (7) give the answer to the question regarding the number of mountains in the continents, which is N_{max} = 3 826, 3 782 and 2 793, respectively (Table 2).

Distribution density of mountains (orosity)

The reader might feel some doubts as to the practical value of knowing the total number of mountains, but it *is* of value as it allows to introduce the term of the mean distribution density of mountains (orosity) of the continents $\langle\bar{\mu}\rangle$:

$$\langle\bar{\mu}\rangle = \frac{N_{max}}{S} = 25.64 \cdot 10^{-6}\ \text{km}^{-2}, \qquad (N_{max} = 3\,823; \quad h_{min} = 600\ \text{m}), \tag{8}$$

here $S = 149 \cdot 10^6$ km^2 is the total area of continents. The orosity of a particular territory may differ from the mean value and is not related to its height above sea level but reflects the roughness of its surface:

$$\begin{cases} \mu >> \langle\bar{\mu}\rangle - \text{mountainous} \\ \mu \approx \langle\bar{\mu}\rangle - \text{hilly} \\ \mu << \langle\bar{\mu}\rangle - \text{plain} \end{cases} \tag{9}$$

Let us consider an example. Nepal is a mountainous country comprising nine of fourteen world's highest mountains; its distribution density of mountains (orosity) is $\mu >> \langle\bar{\mu}\rangle$. Therefore, investments into highway construction do not seem promising in this country. However, in the southern border of Nepal the orosity is $\mu \approx \langle\bar{\mu}\rangle$, i. e. highway building in this part of the country is quite realistic. Furthermore, the fuel consumption while travelling by car is also related to the distribution density of mountains (orosity). Therefore, knowledge of the distribution density of mountains (orosity), alongside the mean height, may be of use in economic assessments.

What is the maximum possible height of a mountain?

Mountains of the continents only are considered in the present study. Yet, one has to realise that high mountains exist also in the oceans. For example, the major part of Mauna Kea mountain (Hawaii, USA) – about 6 000 m – is below sea level. The total height of this mountain is 4 205 + 6 000 = 10 205 m, i. e. it exceeds the height of Everest (8 848 m) by about 1 300 m.

A natural question arises: are there any factors limiting the height of mountains? May the mountains be 20 or 50 km high on the Earth? Could such mountains have existed in the geological past?

Let us present a simple assessment of the maximum height of a mountain. The shear tension σ_r of compressed rock is related to the shear angle θ and the shear modulus of rocks through Hook's law (Kosevich et al., 1986):

$$\sigma_r = G\theta, \tag{10}$$

here $G = \dfrac{E}{2\,(1+h)}$ is the rock shear modulus, E is Young's module, h is Poisson's coefficient. Shear tension is formed by

Table 2. **The number of mountains no less than *h* m high above sea level**
2 lentelė. **Skaičius kalnų, kurių aukštis ne žemesnis nei *h* virš jūros lygio**

η, m	3 500	3 000	2 000	1 000	914.4	610	600
N	534	709	1 277	2 591	2 793	3 782	3 826
δ*N*	47	63	123	342	387	667	682
ε, %	8.85	8.86	9.67	13.2	13.9	17.6	177.8

Notes. N_{th} – the number of mountains the height of which exceeds *h* determined by formula (6); *N* – the real number of mountains from the list (e. g., Scaruffi, 2008); δN_{th} is a theoretical error of the number of mountains according to (7): $\delta N = |N - N_{th}|$ – a real error of the number of mountains; $\varepsilon_{th} = \frac{\delta N_{th}}{N_{th}} \cdot 100\%$ – a theoretical and $\varepsilon = \frac{\delta N}{N} \cdot 100\%$ – a real relative errors per cent.

Pastabos. N_{th} – skaičius kalnų, kurių aukštis viršija *h*, apskaičiuotą pagal formulę (6); *N* – tikrasis kalnų skaičius iš sąrašo (pvz., Scaruffi, 2008); δN_{th} – teorinė kalnų skaičiaus paklaida pagal (7); $\delta N - |N\text{-}N_{th}|$ – tikroji kalnų skaičiaus paklaida; $\varepsilon_{th} = \frac{\delta N_{th}}{N_{th}} \cdot 100\%$ – teorinė ir $\varepsilon = \frac{\delta N}{N} \cdot 100\%$ – tikroji reliatyvioji paklaida procentais.

the huge weight of a mountain, acting on the area S at the foot:

$$\sigma_\tau = \frac{mg}{S} = g\rho h_{\max}, \quad (11)$$

here $g = 9.81$ m/s^2 is free fall acceleration, ρ is rock density, $h_{\max}$ is the mountain's height. Considering that Poisson's coefficient $\mu < 1$ (e. g., for basalt $\mu = 0.22–0.25$ (Cerny et al., 2009) and $G \approx \frac{E}{2} \approx 12.5 \cdot 10^9$ Pa, the density of the compressed matter is higher than in normal conditions, $\rho \approx 10^4$ kg/m^3 and the shear angle $\theta \sim 0.1$ ($\approx 6^o$), from (10) and (11) we obtain a single assessment of the maximum mountain height:

$$h_{\max} = \frac{G\theta}{g\rho} = \frac{12.5 \cdot 10^9\, Pa \cdot 0.1}{10\ \text{m/s}^2 \cdot 10^4 \text{kg/m}^3} = 12.5\ \text{km}. \quad (12)$$

This G height is measured from the foot. Thus, even Mauna Kea (10.2 km) does not exceed this maximum height of 12.5 km.

It should be stressed that the maximum mountain height first of all depends on free acceleration at a similar rock mineral composition and in similar geological conditions. Therefore, e. g., on Mars the maximum mountain height (h_M) should considerably exceed the Earth's mountains (h_E = 8.848 km) due to a lower free acceleration:

$$\frac{h_M}{h_E} \sim \frac{g_E}{g_M}, h_M \sim \frac{g_E}{g_M} h_E \approx 2.63\ h_E, \quad (13)$$

here $g_E = 9.8$ m/s is the free fall acceleration on the Earth's surface and $g_M = 20.38\ g_E$ on the Mars surface. From formula (13) it follows that the highest mountain on Mars could reach about $h_M = 2.63 \cdot h_E \approx 23$ km, which is rather close to the observed values: the highest peak Olympus Mountain on Mars is as high as 21.171 km high with respect to the mean radius of this planet (Rees, 2006). This is the highest known mountain of the Solar system.

Power and exponential approximations

From formula (2) it follows that the distribution of mountains by height is not a scale-invariant dependence, because this property is absent in the exponential function. However, our intuition hints at a similarity. A shift to the logarithmic relations among the heights gives a power function:

$$\ln\left(\frac{h_1}{h}\right) = \beta x^\alpha. \quad (14)$$

If the scale $x \to kx$ and $h \to h^k$, the dependence (14) does not change. This means that the function is scale-invariant and may be related to fractal curves and surfaces.

It, however, does not imply that the authors that used fractality in geology were incorrect. If $\beta x^\alpha << 1$, then it follows from (2) that

$$\frac{h_1}{h(x)} \approx 1 + \beta x^\alpha, \quad (15)$$

i. e. $\frac{h_1}{h-1}$ may be regarded as a scale-invariant value. If $\beta x^\alpha \geq 1$, i. e. $h \leq \frac{h_1}{2}$; the expansion of the approximate exponential function is not accurate. In our case, this should be manifested when $h \leq 4\,424$ m or $n \geq 310$.

This is in accordance with the general rule. If we have a power-type dependence,

$$f(x) = \beta x^\alpha, \quad x \in |0; b|, \quad (16)$$

we may maintain that this dependence is satisfied by experimental data, if the length of the variable interval of x does not exceed b:

$$b = \left(\frac{\ln 2}{\beta}\right)^{\frac{1}{2}}. \quad (17)$$

Otherwise we could mistake this dependence for a very similar, but qualitatively different one, e. g., like in our case, for a power function.

Table 3 presents the standard deviations δN and Pearson's correlation coefficients *Cor* for the observed heights and the heights determined by the mathematical approximating formula, in the first case, according to the power function $a = 4.513 \cdot 10^{-2}$ (these values are indicated in brackets). It should be noted that the proposed mathematical model strongly depends on the number of mountains: the power function of 30 mountains $h(x) = bx^\alpha$ is more precise to represent the decrease in height than the exponential function according to formula (2). However, in a larger interval when $N = 548$, the situation changes: formula (2) is about 591 : 156 ≈ 4 times better to characterize the decrease in height.

Table 3. **Dependence of deviations on the number of mountains *N***
3 lentelė. **Nuokrypių priklausomybė nuo kalnų skaičiaus *N***

N	30	540
δN	75 (51)	156 (591)
Cor	0.97 (0.99)	0.99 (0.91)

Thus, it is not the mountain height ratios but the logarithms of these ratios that are similar, i. e. a small piece of rock only approximately repeats the shape of the mountain; this similarity is more precisely described by the logarithm of the height ratio ln (h_1 / h).

CONCLUSIONS

The main conclusions of the study may be formulated as follows:

1. The height distribution of the highest mountains of the Earth (from 3 500 to 8 848 m) is approximated by the exponential and not the power function.

2. The deviations from the obtained approximation of observed heights of the mountains are close to noise $1 / f^2$.

3. The obtained mathematical approximation allows height assessment for mountains less than 3 500 m high. With the aid of the quantitative definition of the concept of mountain, we assess the total number of mountains on the Earth.

4. The total number of mountains allows introducing the term of distribution density of mountains (orosity) of a geographical region, which can be of use in economic assessments.

5. The maximum possible mountain height on the Earth should not exceed 12.5 km and on Mars 23 km, which is in agreement with the known actual values.

The paper considers the distribution of mountains on the continents. However, the proposed method may be applied to Earth's mountains in general. However, in this case we should account for differences in actual conditions: the pressure at the foot of a mountain, thanks to the effect of the Archimedes law, will be lower if a mountain, or at least part of it, is covered by water. The same applies to submerged mountains.

The author failed to find another theoretical method allowing assessment of the number of mountains more than 2 500 m high.

Thus, the spectrum density function of differences between the theoretical and the observed mountain height distribution is similar to the function $1 / f^2$. However, this method cannot answer the interesting and important question: what is the possible reason for this similarity? This answer seems to be within the competence of rheology and rock material sciences.

Does it mean that B. Mandelbrot was wrong when proposing to model the Earth's surface by fractal multitudes? The results of the present study indicate that Nature and Truth, as always, are more subtle: locally, i. e. within a rather narrow interval, mountain height distribution may be considered as a fractal, but within a wide interval it is not a fractal any more. The situation is similar to that of manifold in the Euclidean space, but globally it is not. A simplified variant of this mathematical statement belongs to Nicolaus Cusano (1401–1464) who maintained that a straight line is part of a very large circle.

Approximation, more complex than regular fractality, is undoubtedly related to the physical properties of geological substances, such as the ratio of Si, Pn metal oxides that determine the density, porosity, elasticity of rocks. Superfractality should manifest itself while studying the global spread of acoustic and electromagnetic waves in rocks. This directly pertains to geomagnetic and seismic phenomena.

Thus, does a small piece of rock resemble the mountain? Yes, it does; more precisely, similar are the dimension ratio logarithms. May they possibly hide in themselves the harmony of the music of mountains?

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KALNŲ AUKŠČIO PASISKIRSTYMO ŽEMĖJE MATEMATINIS MODELIAVIMAS

Santrauka

Aukščiausių Žemės kalnų pasiskirstymo analizė rodo, kad Žemės paviršius gali būti modeliuojamas remiantis matematiniu paviršiumi, kuris yra sudėtingesnis nei įprastas fraktalas ir kurio dimensija nėra pastovus dydis. Nustatyta, kad gautos kalnų aukščio aproksimacinės kreivės nuokrypiai nuo tikrojo kalnų aukščio sudaro statistinį triukšmą, artimą $1 / f^2$. Įvertintas bendras kalnų skaičius ir galimas maksimalus kalno aukštis Žemėje. Įvestas kalnuotumo (angl. *orosity*) terminas, kuris gali būti naudingas darant ekonominius įvertinimus.

Raktažodžiai: kalnai, matematinis modeliavimas, fraktalas